More set up but hopefully it is correct this time!

Let the morphism ϕ be as before contracting the curve E on X.

$$\begin{aligned}\phi : X &\longrightarrow \mathbb{P}^N \\ x &\longmapsto [s_0(x) : s_1(x) : \ldots : s_N(x)]\end{aligned}$$

Objective: We would like to show that the image $\phi(X) = Y$ is smooth at $p = \phi(E)$. To do this we must consider the local ring $\mathcal{O}_{Y,p}$.

<u>Fact</u> $\phi_*\mathcal{O}_X = \mathcal{O}_Y$. This follows from Y being normal in a *Zariski's Main Theorem* like argument.

Let $V = X_0 \neq 0$ the affine open set on $\mathbb{P}^N$ and let $U \subset X$ be the open set $\phi^{-1}(V) = \{s_0 \neq 0\}$. Then

$$\begin{aligned}\mathcal{O}_Y(V) &= \phi_*\mathcal{O}_X(V) \\ &= \mathcal{O}_X(U)\end{aligned}$$

And the local ring

$$\begin{aligned}\mathcal{O}_{Y,p} &= \lim_{W \ni p} \mathcal{O}_Y(W) \\ &= \lim_{W \subset E} \mathcal{O}_X(W) \\ &=: \Gamma(E, \mathcal{O}_X)\end{aligned}$$

Where the inverse limit is indexed over open sets W. The sections $\Gamma(E, \mathcal{O}_X)$ can be thought of as an equivalence relation $\{(\tau, W) \mid \tau \in \mathcal{O}_X(W),\ W \text{ is open}\}$ where $(\tau, W) \sim (\tau', W')$ iff

$$\exists W'' \text{ such that } \tau|_{W''} = \tau'|_{W''}$$

The maximal ideal $\mathfrak{m} = (X_1/X_0, X_2/X_0, \ldots, X_N/X_0)$ define the point $p \in Y$ in the affine coordinate set V. We need look at the image of $\mathfrak{m}$ in $\O_{Y,p}$, $\overline{\mathfrak{m}}$, and compute

$$\dim \left(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2\right)^\vee = \dim T_pY$$

New Objective Show that $\dim \left(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2\right) = 2$

$\overline{(m)}$ has the generators $\overline{X_i/X_0}$ where these are the elements the images of X_i/X_0 under the canonical map

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_{Y,p}$$

taking sections to their germs. By the isomorphism of sheaves $\phi_*(\mathcal{O}_X) \cong \mathcal{O}_Y$ we can consider the maximal ideal $\mathfrak{n} = (s_1/s_0, ..., s_n/s_0) \lhd \mathcal{O}_X(U)$ and it's image $\overline{\mathfrak{n}}$ under the map

$$\mathcal{O}_X(U) \longrightarrow \Gamma(E, \mathcal{O}_X)$$

the generators of which are given by the germs $(\overline{s_i}/\overline{s_0}, U)$ where $U = \{s_0 \neq 0\}$.

Final Objective We would like to show that

$$\dim(\overline{\mathfrak{n}}/\overline{\mathfrak{n}}^2) = 2$$

with generators given by $\overline{s_1}/\overline{s_0}$ and $\overline{s_2}/\overline{s_0}$

A useful thing There is an isomorphism

$$H^0(U, \mathcal{O}_X(D)) \underset{1/s_0\cdot}{\overset{s_0\cdot}{\rightleftarrows}} H^0(U, \mathcal{O}_X)$$

induced by multiplication of the sections s_0 and $1/s_0$. This isomorphism holds for all open sets W, $E \subset W \subseteq U$ so this isomorphisms descends under the inverse limit to give

$$\Gamma(U, \mathcal{O}_X(D)) \underset{1/s_0\cdot}{\overset{s_0\cdot}{\rightleftarrows}} \Gamma(U, \mathcal{O}_X)$$

So I can work with set generators $\{\overline{s_1}, \ldots, \overline{s_N}\}$ in $\Gamma(E, \mathcal{O}_X(D))$ too.

Strategy for proof

??*passing to the local ring $\mathcal{O}_{Y,p}$ we can assume that the vector space $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is finite dimensional*??

Working now in in $\Gamma(E, \mathcal{O}_X)$ we can take a generator $\overline{s_i}/\overline{s_0}$, it must vanish to some order along E. We can 'modify' it with a polynomial in $\overline{s_1}/\overline{s_0}$ and $\overline{s_2}/\overline{s_0}$ to produce an element that vanishes to arbitrary order along E. *Since $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is finite dimensional*, we must have 'modified' $\overline{s_i}/\overline{s_0}$ so that it is now in the $\overline{\mathfrak{m}}^2$.

This is how the modification procedure takes place: We have shown that the group elements of

$$V(U, n) := H^0(U, \mathcal{O}(D - nE))/H^0(U, D - (n+1)E)$$

are generated by elements in $V(U, 1)$. The elements in $V(U, 1)$ are generated by the images of s_1, s_2 under the quotient

$$H^0(U, \mathcal{O}_X(D - E)) \longrightarrow H^0(U, \mathcal{O}_X(D - E))/H^0(U, \mathcal{O}_X(D - 2E))$$

Suppose that s_i vanishes along E to order $m > 1$ then we can express

$$s_i = \alpha \cdot s_1 + \beta \cdot s_2$$

in the group $V(U, m)$ where α and β are polynomials in s_1/s_0 and s_2/s_0 with degree at $m-1$. Hence

$$s_i - \alpha \cdot s_1 - \beta \cdot s_2 \in H^0(U, \mathcal{O}_X(D - (m+1)E)).$$

Now this new section must vanish to some order $m' > m$ and we repeat the procedure to find polynomials α_N and β_N such that

$$s_i - \alpha_N \cdot s_1 - \beta_N \cdot s_2 \in H^0(U, \mathcal{O}_X(D - N \cdot E)$$

to some large positive integer N.

We can then pass on to the ring $\Gamma(E, \mathcal{O}_X)$ by the composition

$$H^0(U, \mathcal{O}_X(D)) \longrightarrow \Gamma(E, \mathcal{O}_X(D)) \xrightarrow{1/\overline{s_0}} \Gamma(E, \mathcal{O}_X)$$

to find that

$$\overline{s_i}/\overline{s_0} - P(\overline{s_1}/\overline{s_0}, \overline{s_2}/\overline{s_0})$$

vanishes to some large order N along E,where

$$P(\overline{s_1}/\overline{s_0}, \overline{s_2}/\overline{s_0} = \overline{\alpha_N} \cdot \overline{s_1}/\overline{s_0} - \overline{\beta_N} \cdot \overline{s_2}/\overline{s_0}.$$

By the finiteness assumption we should in fact have that

$$\overline{s_i}/\overline{s_0} - P(\overline{s_1}/\overline{s_0}, \overline{s_2}/\overline{s_0}) \in \overline{\mathfrak{n}}^2$$