universität freiburg

Sur les assistants de preuve

Prof. Dr. Stefan Kebekus
Freiburg, 29. September 2025

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Sur les assistants de preuve ...

... for the Working Mathematician

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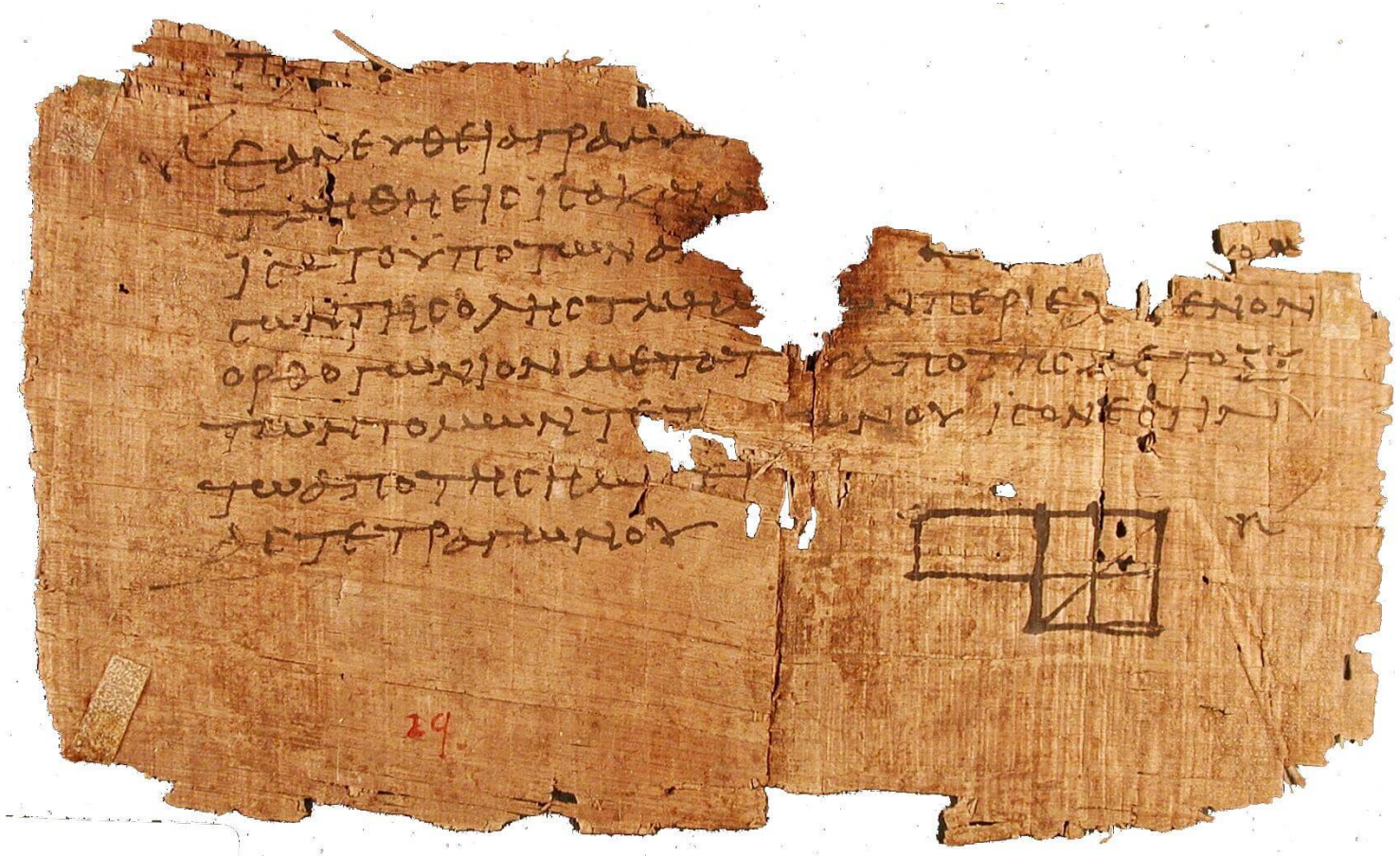
Agenda

1. Background
2. Practical Example
3. State of the Art
4. Why Formalize Mathematics?
5. Outlook

Background

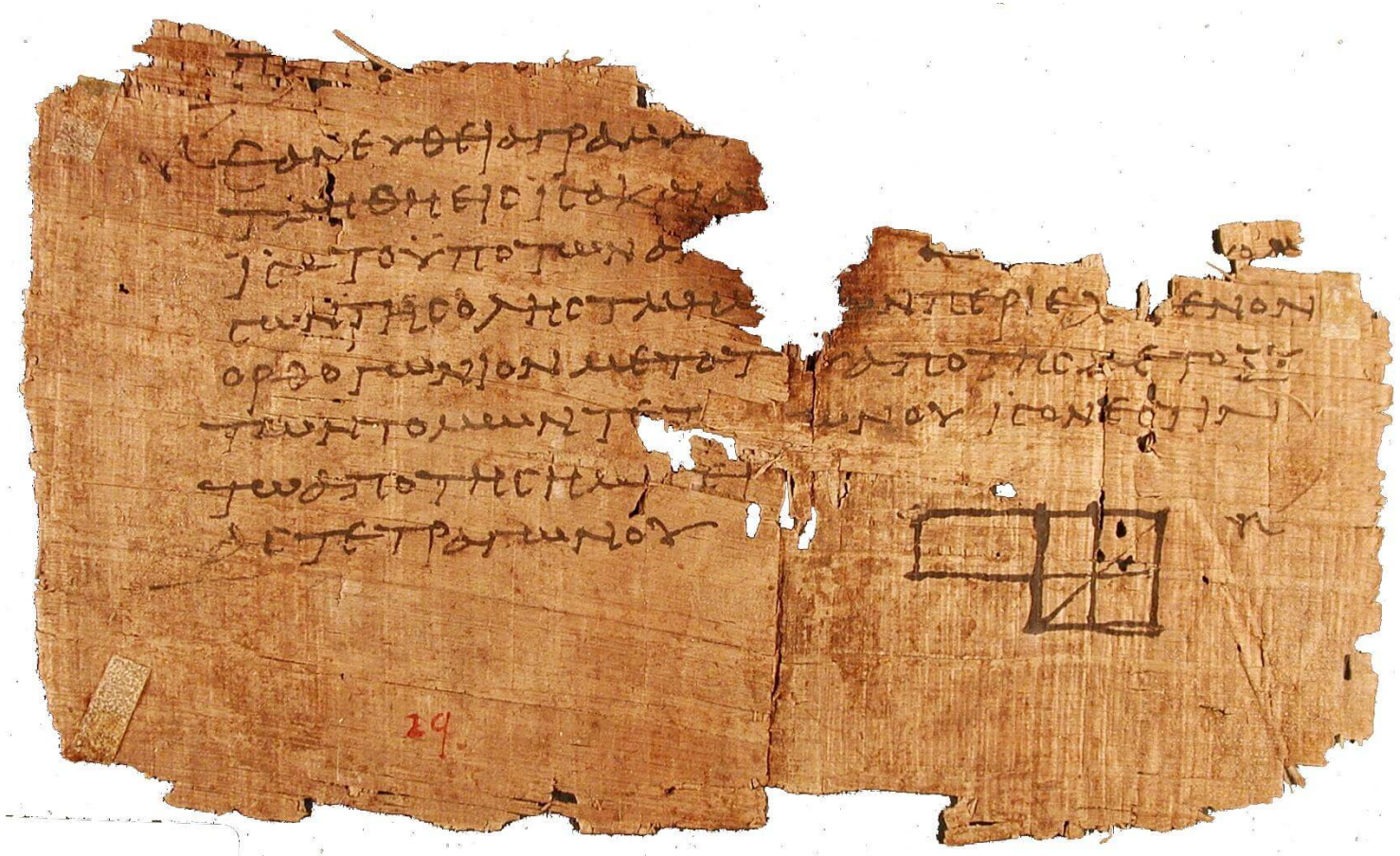
Background

Στοιχεῖα



Background

Euclid's „Elements“



Background
Euclid's „Elements“

- **Introduces the axiomatic method**
  - Specifies small number of axioms
  - Builds Mathematics from the axioms through logical inference
- **Widely recognized and taught as the „correct“ way to do mathematics**

Background
Euclid's „Elements“

- **Introduces the axiomatic method**
  - Specifies small number of axioms
  - Builds Mathematics from the axioms through logical inference
- **Widely recognized and taught as the „correct“ way to do mathematics**
- **Widely ignored by the working mathematician in practise**

Background
Principia Mathematica

$*54{\cdot}43. \quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Background
Curry-Howard Correspondence

There is a direct relationship between logical propositions and types in programming languages, and between mathematical proofs and computer programs.

Core Idea:

- Propositions correspond to types in programming.
- Proofs of a proposition correspond to programs that construct elements of the type corresponding to the proposition.
- Proving a proposition corresponds to writing a program.

Background
Curry-Howard Correspondence

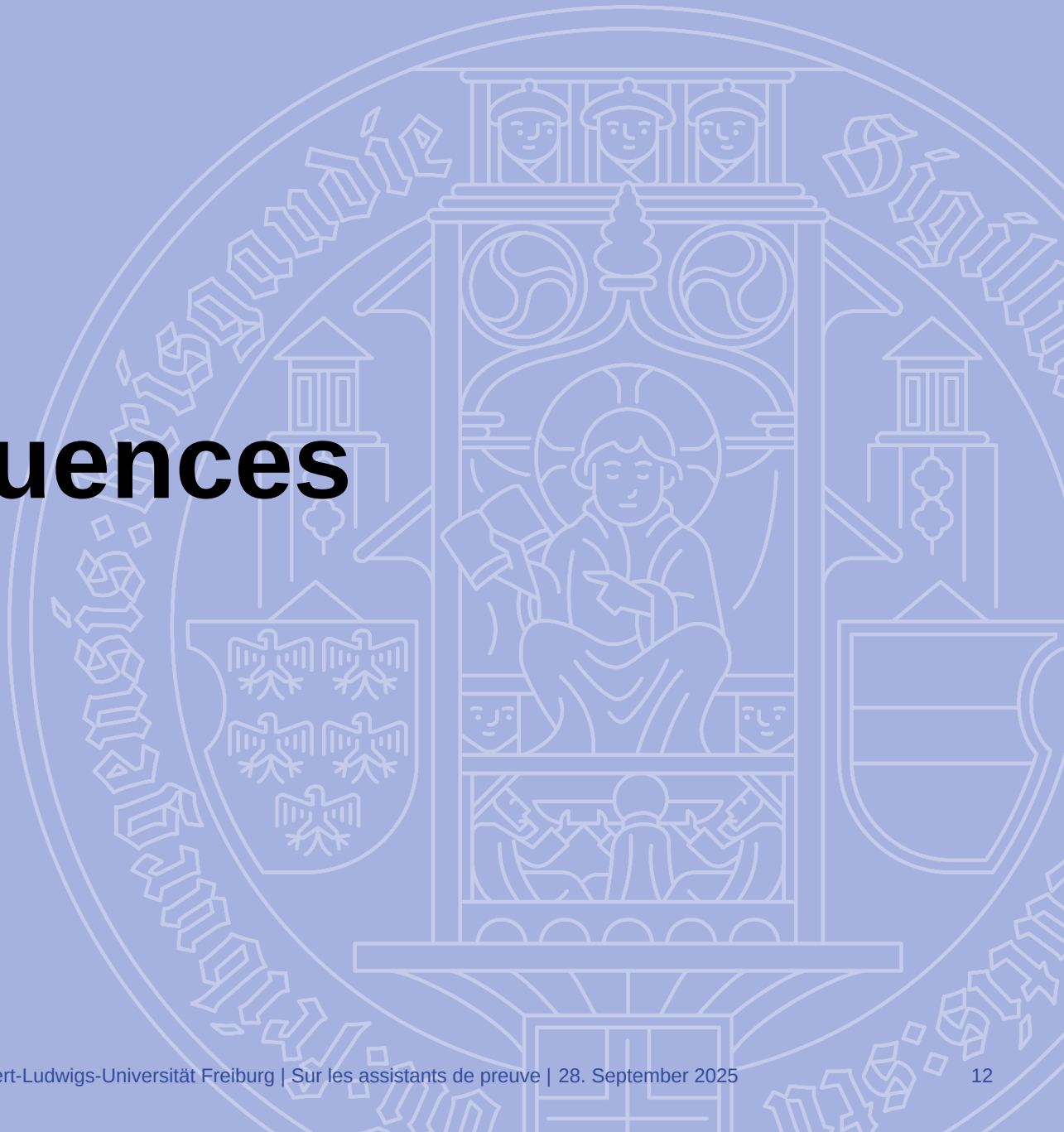
There is a direct relationship between logical propositions and types in programming languages, and between mathematical proofs and computer programs.

Want a programming language where …

- types already „look like“ their corresponding propositions in standard mathematical language, and
- programs already „look like“ their corresponding proofs in standard mathematical language.

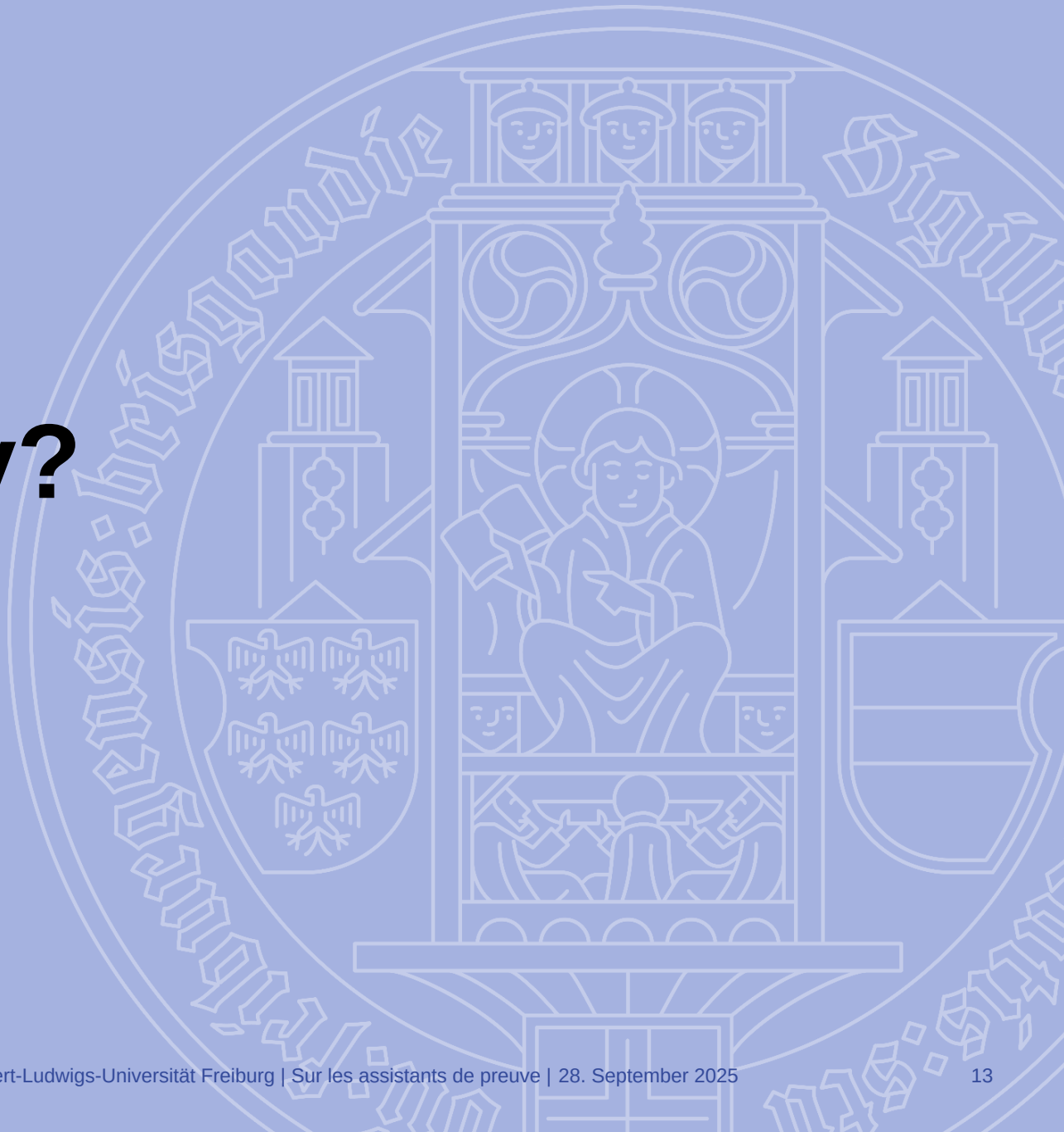
Key Point: Computers can check if source code contains a valid program!

Practical Example
Continuity and Sequences



State of the Art

Where are we today?



State of the Art
Formalization Projects / Research

Reference Projects

- Thomas Hales: Flyspec / Kepler Conjecture (Completed 2015)
- Johan Commelin: Liquid Tensor Experiment / Condensed Mathematics (Completed 2021)
- Kevin Buzzard: Fermat‘s Last Theorem (Ongoing)

Other

- Smale Sphere Eversion / Independence of the Continuum Hypothesis / Banach-Tarski Paradox

Massive push by high-profile mathematicians including Timothy Govers, Terence Tao, Akshay Venkatesh, Maryna Viazovska, …

State of the Art
Mathlib

Goal: Bourbaki in the 21st Century / Building the Mathematics Library of the Future

- Base project of most formalization efforts today
- Huge community / 500+ Contributors
- Math coverage presently uneven, but entire curriculum of Imperial College London is within reach

State of the Art
Formalization Projects / Own Experience

Goal:

- Formalize Value Distribution Theory
- Contribute the resulting code to Mathlib

Experience:

- Steep learning curve, but excellent tutorials available
- Extremely helpful and welcoming community
- Tedious work, but no fundamental problems

State of the Art
Education

Integration into the curriculum

- Discussed and (partially) implemented in numerous institutions all over the globe
- Good ressources and materials become available
- Naive implementation likely to overwhelm students

Formalization as a thesis project

- At Imperial College London: 20-30% of all MA/BA thesis projects are concerned with formalization

State of the Art
Governance and Development

LEAN

- Open Source / Apache License
- Initially developed by Microsoft Research
- Today: Development through FRO financed by US and British Philantropists

Mathlib

- Open Source / Apache License
- Development overseen by small group of extremely competent volunteer maintainers
- Developmend supported by the Mathlib Foundation, financed by US and British Philantropists
- Infrastructure financed by GitHub Inc. / Microsoft

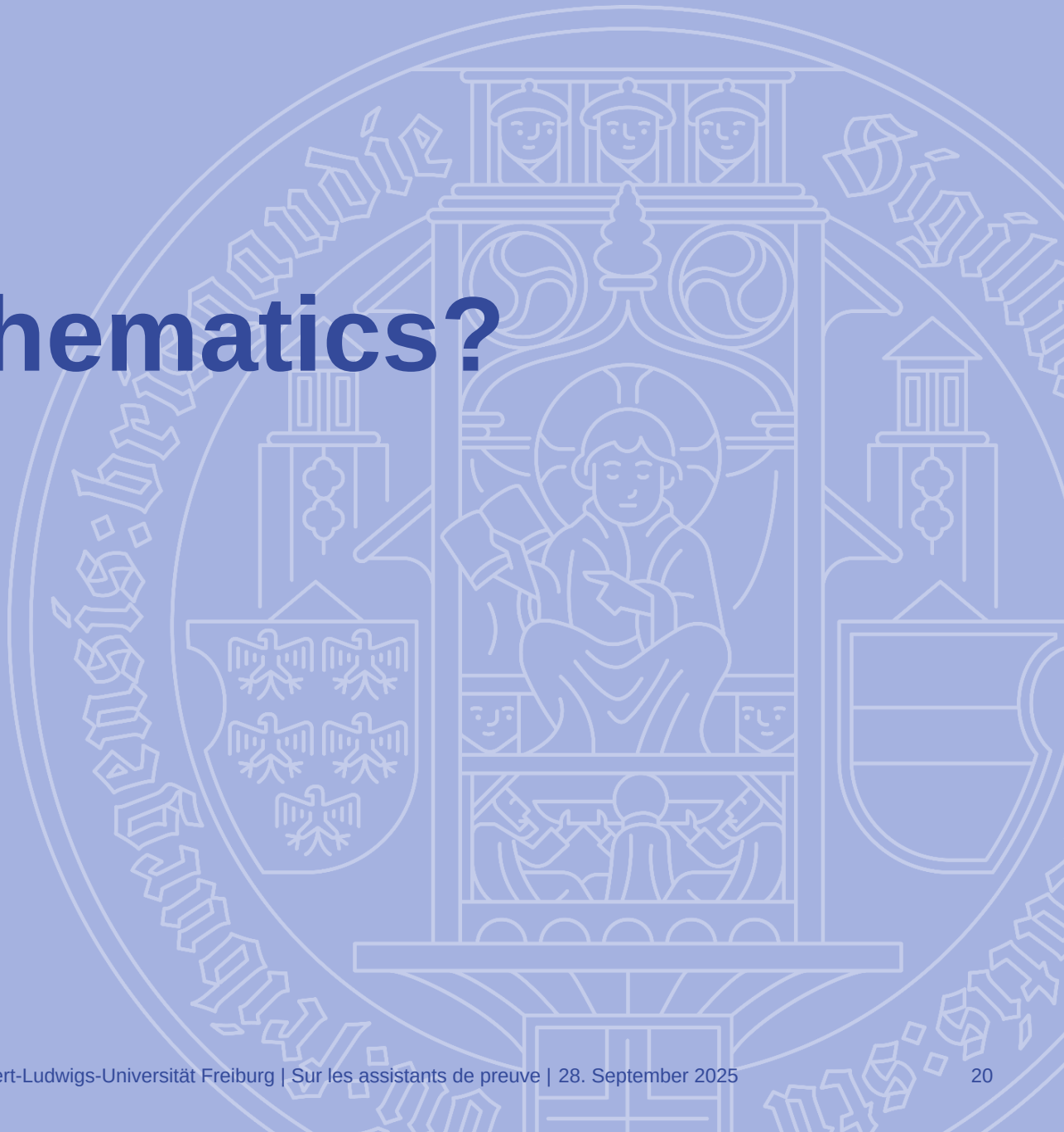
State of the Art
Artificial Intelligence / Automatic Formalization

- Performance currently underwhelming, not good enough for daily use
- Overblown claims targeted at potential investors, substantial marketing hyperbole
- Real improvements in the last two years

Massive Investments

- US Cooperations: OpenAI / Microsoft / Google / Meta / …
- US Venture Capital via AI Startups
- DARPA
- US and British Philantropists

Why Formalize Mathematics?



Why Formalize Mathematics?

- **For fun!**
- **To make knowledge available!**
- **To preserve knowledge!**

- **As a direct tool in research.**
- **To verify proofs.**

Outlook

Where are we going?

Thank you for your time!